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A Chebyshev type inequality for Sugeno integral and comonotonicity

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ABSTRACT

We supply a characterization of comonotonicity property by a Chebyshev type inequality for Sugeno integral.

1. Introduction and preliminaries

Given a measurable space $(\Omega, \mathcal{F})$ and two $\mathcal{F}$-Borel measurable functions X, Y, Armstrong [2] proves that, if X, Y are Lebesgue integrable w.r.t. a probability measure P and are comonotonic, then the following well known *Chebyshev inequality*:

$$\int_\Omega XY\,d\mathrm{P} \geqslant \left(\int_\Omega X\,d\mathrm{P}\right)\left(\int_\Omega Y\,d\mathrm{P}\right) \tag{1}$$

holds; conversely, if (1) holds for any probability measure P, then X, Y are comonotonic. Consequently, keeping in mind the linearity of Lebesgue integral w.r.t. positive linear combinations of measures, we have that, if X, Y are Lebesgue integrable w.r.t. a real measure μ and are comonotonic, then the following modified Chebyshev inequality:

$$\|\mu\| \int_\Omega XY\,d\mu \geqslant \left(\int_\Omega X\,d\mu\right)\left(\int_\Omega Y\,d\mu\right) \tag{2}$$

holds (with $\|\mu\| = \mu(\Omega)$); conversely, if (2) holds for any real measure μ, then X, Y are comonotonic.

In this paper, motivated by this result and the growing interest in studying the validity of (1) for Sugeno integral in the setting of fuzzy measures (see [4,6] and see [1,5] for a generalization in which a suitable operation on $[0,+\infty[^2$ is considered in (1) instead of the product operation), we prove that X, Y are comonotonic iff the following version of Chebyshev inequality:

$$\|\mu\| \mathrm{S}\int_\Omega \frac{1}{\|\mu\|} XY\,d\mu \geqslant \left(\mathrm{S}\int_\Omega X\,d\mu\right)\left(\mathrm{S}\int_\Omega Y\,d\mu\right) \tag{3}$$

holds for any real non-null monotone set function (not necessarily upper and/or lower continuous) μ on $\mathcal{F}$, where $\mathrm{S}\int$ denotes the Sugeno integral. We also supply an extension of theorems 2.6, 2.7 in Ouyang et al. [6].

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Now, we are going to recall briefly some notion and notation useful in the sequel. We shall denote by ω (with or without indices) any element of Ω. Any set function $\mu : \mathcal{F} \to [0, +\infty]$ is called a *monotone set function* if the following properties are satisfied:

(a) $\mu(\emptyset) = 0$;
(b) $F_1 \subset F_2$ implies $\mu(F_1) \leqslant \mu(F_2)$ (monotonicity);
moreover, μ is called *real* if $\|\mu\| = \mu(\Omega) < +\infty$.

Given a real $k > 0$, we denote by k_ω the 0–k *valued Dirac measure at* ω (i.e. $k_\omega(F) = k$, if $\omega \in F$, and $k_\omega(F) = 0$, if $\omega \notin F$).

Henceforth, X, Y, Z always denote non-negative real-valued functions on Ω which are $\mathcal{F}$-Borel measurable. Moreover, given X, we put $\{X \geqslant \alpha\} = \{\omega : X(\omega) \geqslant \alpha\}$ for any $\alpha \geqslant 0$. Finally, we recall that X, Y are said to be *comonotonic* if $X(\omega_1) > X(\omega_2)$ and $Y(\omega_1) < Y(\omega_2)$ is impossible for any ω_1, ω_2 (i.e. $(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geqslant 0$ for any ω_1, ω_2).

Now, given the monotone set function space $(\Omega, \mathcal{F}, \mu)$, the *Sugeno integral* of X w.r.t. μ is defined as:

$$\mathrm{S}\int_\Omega X\,d\mu = \bigvee_{\alpha \geqslant 0} [\alpha \wedge \mu(X \geqslant \alpha)] := \sup_{\alpha \geqslant 0} \min(\alpha, \mu(\{X \geqslant \alpha\})).$$

2. Main results

We start with some lemmas. Note that the equivalence (i) $\Leftrightarrow$ (ii) in the next lemma is proved in Wang and Klir [7] for fuzzy measures.

Lemma 2.1. *Given $\beta > 0$, the following statements are equivalent:*

(i) $\mathrm{S}\int_\Omega X\,d\mu \geqslant \beta$;
(ii) $\mu(X \geqslant \alpha) \geqslant \beta$, *for any* $\alpha < \beta$;
(iii) *There is a strictly increasing sequence* $(\alpha_n)_{n \geqslant 1}$ *such that* $\alpha_n \uparrow \beta$ *and* $\mu(X \geqslant \alpha_n) \geqslant \beta$ *for any* n.

Proof. (i) $\Rightarrow$ (ii) Assume (ii) is not true, i.e. there is $\alpha < \beta$ such that $\mu(X \geqslant \alpha) < \beta$. Then, $\delta = \alpha \vee \mu(X \geqslant \alpha) < \beta$, so that (recall $\mu(X \geqslant \gamma)$ is decreasing w.r.t. γ)

$$\mathrm{S}\int_\Omega X\,d\mu = \bigvee_{\gamma \geqslant 0} [\gamma \wedge \mu(X \geqslant \gamma)] = \bigvee_{\gamma \in [0,\delta[} [\gamma \wedge \mu(X \geqslant \gamma)] \vee \bigvee_{\gamma \geqslant \delta} [\gamma \wedge \mu(X \geqslant \gamma)] \leqslant \delta \vee \mu(X \geqslant \delta) \leqslant \delta \vee \mu(X \geqslant \alpha) = \delta < \beta.$$

(iii) $\Rightarrow$ (i) It is enough to observe that $\mathrm{S}\int_\Omega X\,d\mu \geqslant \alpha_n \wedge \mu(X \geqslant \alpha_n) \geqslant \alpha_n \wedge \beta = \alpha_n$ for any n. This completes the proof. $\square$

In the next lemma we supply some rules of calculus for Sugeno integral.

Lemma 2.2. *The following statements hold:*

(i) $\mathrm{S}\int_\Omega X\,d(k\mu) = k\,\mathrm{S}\int_\Omega \frac{1}{k}X\,d\mu$, *for any real* $k > 0$;
(ii) *Let* $\ell \geqslant \mathrm{S}\int_\Omega X\,d\mu$. *Then,* $\mathrm{S}\int_\Omega X\,d\mu^{(\ell)} = \mathrm{S}\int_\Omega X\,d\mu$, *where* $\mu^{(\ell)}(F) = \ell \wedge \mu(F)$ *for any* $F \in \mathcal{F}$;
(iii) *Given real numbers* $a, b > 0$, *let* ω_1, ω_2 *and* Z *be such that* $Z(\omega_1) \leqslant Z(\omega_2)$. *Then,* $\mathrm{S}\int_\Omega Z\,d(a_{\omega_1} + b_{\omega_2}) = [Z(\omega_1) \wedge (a+b)] \vee [Z(\omega_2) \wedge b]$.

Proof

(i) Given a real number $k > 0$, we have

$$\mathrm{S}\int_\Omega X\,d(k\mu) = \bigvee_{\alpha \geqslant 0} [\alpha \wedge k\mu(X \geqslant \alpha)] = k \bigvee_{\alpha \geqslant 0} \left[\frac{\alpha}{k} \wedge \mu\left(\frac{1}{k}X \geqslant \frac{\alpha}{k}\right)\right] = k \bigvee_{\beta \geqslant 0} \left[\beta \wedge \mu\left(\frac{1}{k}X \geqslant \beta\right)\right] = k\,\mathrm{S}\int_\Omega \frac{1}{k}X\,d\mu.$$

(ii) On noting that $\mu^{(+\infty)} = \mu$, assume $\ell < +\infty$. Given $\alpha \geqslant 0$, from $\ell \geqslant \mathrm{S}\int_\Omega X\,d\mu \geqslant \alpha \wedge \mu(X \geqslant \alpha)$ we have $\alpha \wedge \mu(X \geqslant \alpha) = \ell \wedge (\alpha \wedge \mu(X \geqslant \alpha)) = \alpha \wedge (\ell \wedge \mu(X \geqslant \alpha)) = \alpha \wedge \mu^{(\ell)}(X \geqslant \alpha)$. Therefore, $\mathrm{S}\int_\Omega X\,d\mu^{(\ell)} = \mathrm{S}\int_\Omega X\,d\mu$.
(iii) It is enough to observe that

$$(a_{\omega_1} + b_{\omega_2})(Z \geqslant \alpha) = \begin{cases} a+b & \text{if } \alpha \leqslant Z(\omega_1) \\ b & \text{if } Z(\omega_1) < \alpha \leqslant Z(\omega_2) \\ 0 & \text{if } \alpha > Z(\omega_2). \end{cases}$$

This completes the proof. $\square$

The following lemma assures that the well known change of variables rule holds for Sugeno integral, as well.

Lemma 2.3. *Let $T:\Omega \to [0,+\infty[$ be a $\mathcal{F}$-Borel measurable function and I an interval including $T(\Omega)$. Define the monotone set function $\mu_T = \mu T^{-1}$ as:*

$$\mu_T(B) = \mu(T^{-1}(B))$$

for any Borel set B in I. Then, for any Borel measurable function $f: I \to [0,+\infty[$, we have:

$$\mathrm{S}\int_\Omega f(T)\,d\mu := \mathrm{S}\int_\Omega f\circ T\,d\mu = \mathrm{S}\int_I f\,d\mu_T.$$

Proof. We have

$$\mathrm{S}\int_I f\,d\mu_T = \bigvee_{\alpha\geqslant 0}[\alpha\wedge\mu_T(f\geqslant\alpha)] = \bigvee_{\alpha\geqslant 0}\left[\alpha\wedge\mu(T^{-1}(f\geqslant\alpha))\right] = \bigvee_{\alpha\geqslant 0}[\alpha\wedge\mu(f\circ T\geqslant\alpha)] = \mathrm{S}\int_\Omega f(T)\,d\mu.$$

This completes the proof. □

Now, we are able to prove, for comonotonic functions, two Chebyshev type inequalities for Sugeno integral w.r.t. a real non-null monotone set function μ (without any lower continuity and/or upper continuity assumption on μ).

Theorem 2.4 (*Main theorem*). *Let X, Y be comonotonic. Then, the following inequalities*

$$(\|\mu\|\vee 1)\mathrm{S}\int_\Omega XY\,d\mu \geqslant \|\mu\|\mathrm{S}\int_\Omega \frac{1}{\|\mu\|}XY\,d\mu \geqslant \left(\mathrm{S}\int_\Omega X\,d\mu\right)\left(\mathrm{S}\int_\Omega Y\,d\mu\right)$$

hold for any real non-null monotone set function μ.

Proof. Let $0 < \|\mu\| < +\infty$. Note that the former inequality does not depend on the comonotonicity hypothesis. Indeed, if $\|\mu\| \leqslant 1$, by Lemma 2.2(i), we have

$$\|\mu\|\mathrm{S}\int_\Omega \frac{1}{\|\mu\|}XY\,d\mu = \mathrm{S}\int_\Omega XY\,d(\|\mu\|\mu) \leqslant \mathrm{S}\int_\Omega XY\,d\mu,$$

on noting that $\|\mu\|\mu \leqslant \mu$. If $\|\mu\| > 1$, then $\frac{1}{\|\mu\|}XY \leqslant XY$ and hence, by monotonicity of Sugeno integral, $\mathrm{S}\int_\Omega XY\,d\mu \geqslant \mathrm{S}\int_\Omega \frac{1}{\|\mu\|}XY\,d\mu$.

In order to verify the latter inequality, let X, Y be comonotonic. The proof is carried out in two steps.

1°. Let $\|\mu\| = 1$. By putting $T = X + Y \geqslant 0$, there are, by Proposition 4.5(v) in Denneberg [3], two continuous increasing functions $f_1, f_2: I \to [0,+\infty[$ such that $I = [\inf T(\Omega),+\infty[$ and $X = f_1(T)$, $Y = f_2(T)$. Consequently, by Lemma 2.3, we have

$$\mathrm{S}\int_\Omega X\,d\mu = \mathrm{S}\int_I f_1\,d\mu_T, \quad \mathrm{S}\int_\Omega Y\,d\mu = \mathrm{S}\int_I f_2\,d\mu_T,$$
$$\mathrm{S}\int_\Omega XY\,d\mu = \mathrm{S}\int_\Omega (f_1f_2)\circ T\,d\mu = \mathrm{S}\int_I f_1f_2\,d\mu_T.$$

Now, let

$$a_i = \mathrm{S}\int_I f_i\,d\mu_T \leqslant \|\mu_T\| = 1 \quad (i = 1,2) \tag{4}$$

and suppose $a_1, a_2 > 0$ (otherwise, the inequality is trivial). Given a natural number n, from

$$\left\{f_1f_2 \geqslant \left(a_1-\frac{1}{n}\right)\left(a_2-\frac{1}{n}\right)\right\} \supset \left\{f_1 \geqslant a_1-\frac{1}{n}\right\} \cap \left\{f_2 \geqslant a_2-\frac{1}{n}\right\}$$

we get

$$\mu_T\left(f_1f_2 \geqslant \left(a_1-\frac{1}{n}\right)\left(a_2-\frac{1}{n}\right)\right) \geqslant \mu_T\left(\left\{f_1 \geqslant a_1-\frac{1}{n}\right\} \cap \left\{f_2 \geqslant a_2-\frac{1}{n}\right\}\right) = \min_{i=1,2}\mu_T\left(f_i \geqslant a_i-\frac{1}{n}\right),$$

where the equality holds since the intersection is equal to one of the two sets, on noting that f_i is increasing on $I (i = 1,2)$. Consequently, recalling (4), by Lemma 2.1 ((i) ⇒ (ii) with $\beta = a_i$ and $\alpha = a_i - \frac{1}{n}$), we get

$$\mu_T\left(f_1f_2 \geqslant \left(a_1-\frac{1}{n}\right)\left(a_2-\frac{1}{n}\right)\right) \geqslant \min(a_1,a_2) \geqslant a_1a_2.$$

Finally, by Lemma 2.1 ((iii) ⇒ (i)), we have

$$\mathrm{S}\int_I f_1 f_2\, d\mu_T \geqslant a_1 a_2 = \left(\mathrm{S}\int_I f_1\, d\mu_T\right)\left(\mathrm{S}\int_I f_2\, d\mu_T\right),$$

so that the desired inequality immediately follows.

2°. Let $\|\mu\| \neq 1$ and $\mu' = \frac{1}{\|\mu\|}\mu$. Then, by Lemma 2.2(i) and step 1° (note that $\frac{1}{\|\mu\|}X,\ \frac{1}{\|\mu\|}Y$ are comonotonic), we have

$$\begin{aligned}\|\mu\|\mathrm{S}\int_\Omega \frac{1}{\|\mu\|}XY\, d\mu &= \|\mu\|\mathrm{S}\int_\Omega \frac{1}{\|\mu\|}XY\, d(\|\mu\|\mu') = \|\mu\|^2\,\mathrm{S}\int_\Omega \left(\frac{1}{\|\mu\|}X\right)\left(\frac{1}{\|\mu\|}Y\right)d\mu' \\ &\geqslant \|\mu\|^2\left(\mathrm{S}\int_\Omega \frac{1}{\|\mu\|}X\, d\mu'\right)\left(\mathrm{S}\int_\Omega \frac{1}{\|\mu\|}Y\, d\mu'\right) = \left(\|\mu\|\mathrm{S}\int_\Omega \frac{1}{\|\mu\|}X\, d\mu'\right)\left(\|\mu\|\mathrm{S}\int_\Omega \frac{1}{\|\mu\|}Y\, d\mu'\right) \\ &= \left(\mathrm{S}\int_\Omega X\, d\mu\right)\left(\mathrm{S}\int_\Omega Y\, d\mu\right).\end{aligned}$$

This completes the proof. □

Remark 2.5

(i) The Chebyshev type inequality

$$(\|\mu\| \vee 1)\mathrm{S}\int_\Omega XY\, d\mu \geqslant \left(\mathrm{S}\int_\Omega X\, d\mu\right)\left(\mathrm{S}\int_\Omega Y\, d\mu\right)$$

considered in main theorem, becomes (1), if $\|\mu\| \leqslant 1$, and (2), if $\|\mu\| > 1$.

(ii) We supply two examples in order to show that both inequalities considered in the previous theorem may be strict, either $\|\mu\| < 1$ or $\|\mu\| > 1$. Let Ω be an interval of the real line, $\mathcal{F}$ the Borel σ-field on Ω and λ the Lebesgue measure on $\mathcal{F}$. Moreover, let $X(\omega) = \omega$ and $Y(\omega) = 2\omega$ for any ω. Now, let $\Omega = [0,3]$ and $\mu = \lambda$. Then, by a straightforward calculus, we get $\mathrm{S}\int_\Omega X\, d\mu = \frac{3}{2},\ \mathrm{S}\int_\Omega Y\, d\mu = 2$ and $\mathrm{S}\int_\Omega \frac{1}{\|\mu\|}XY\, d\mu = \mathrm{S}\int_\Omega \frac{1}{3}XY\, d\mu = \frac{3}{2},\ \mathrm{S}\int_\Omega XY\, d\mu = 2$. Finally, consider $\Omega = [0,3]$ and $\mu = \frac{1}{2}\lambda$. Then, by a straightforward calculus, we get $\mathrm{S}\int_\Omega X\, d\mu = \frac{1}{3},\ \mathrm{S}\int_\Omega Yd\mu = \frac{2}{5}$ and $\mathrm{S}\int_\Omega \frac{1}{\|\mu\|}XY\, d\mu = \mathrm{S}\int_\Omega 2XY\, d\mu = \frac{17-\sqrt{33}}{32},\ \mathrm{S}\int_\Omega XY\, d\mu = \frac{9-\sqrt{17}}{16}$.

The following corollary generalizes Theorems 2.6 and 2.7 in Ouyang et al. [6].

Corollary 2.6. *Let X, Y be comonotonic and* $\mathrm{S}\int_\Omega X\, d\mu \leqslant 1,\ \mathrm{S}\int_\Omega Y\, d\mu \leqslant 1$. *Then,*

$$\mathrm{S}\int_\Omega XY\, d\mu \geqslant \left(\mathrm{S}\int_\Omega X\, d\mu\right)\left(\mathrm{S}\int_\Omega Y\, d\mu\right).$$

Proof. Assume $\|\mu\| > 0$. Then, by main theorem and Lemma 2.2(ii), we have

$$\mathrm{S}\int_\Omega XY\, d\mu^{(1)} = (\|\mu^{(1)}\| \vee 1)\mathrm{S}\int_\Omega XY\, d\mu^{(1)} \geqslant \left(\mathrm{S}\int_\Omega X\, d\mu^{(1)}\right)\left(\mathrm{S}\int_\Omega Y\, d\mu^{(1)}\right) = \left(\mathrm{S}\int_\Omega X\, d\mu\right)\left(\mathrm{S}\int_\Omega Y\, d\mu\right).$$

Consequently, on noting that $\mu \geqslant 1 \wedge \mu = \mu^{(1)}$, the thesis immediately follows from $\mathrm{S}\int_\Omega XY\, d\mu \geqslant \mathrm{S}\int_\Omega XY\, d\mu^{(1)}$. □

Finally, we are going to prove our characterization of comonotonicity via Chebyshev type inequality (3).

Theorem 2.7 (*Characterization theorem*). *The following statements are equivalent:*

(i) *X, Y are comonotonic;*
(ii) *The inequality (3) holds for any real non-null monotone set function* μ;
(iii) *The inequality (3) holds for any uniform two-points measure* μ *(i.e.* $\mu = a_{\omega_1} + a_{\omega_2}$, *with* $a > 0$ *and* $\omega_1 \neq \omega_2$*).*

Proof. By main theorem, it is enough to verify only (iii) ⇒ (i). Assume X, Y to be non comonotonic, i.e. there are ω_1, ω_2 such that $(X(\omega_2) - X(\omega_1))(Y(\omega_2) - Y(\omega_1)) < 0$. Without loss of generality, suppose $X(\omega_1) < X(\omega_2)$ and $Y(\omega_1) > Y(\omega_2)$. Now, let $a \geqslant \max\{X(\omega_i), Y(\omega_i): i = 1,2\}$ and $\mu = a_{\omega_1} + a_{\omega_2}$. Then, by Lemma 2.2(iii), we have $\mathrm{S}\int_\Omega X\, d\mu = X(\omega_2),\ \mathrm{S}\int_\Omega Y\, d\mu = Y(\omega_1)$ and

$$\|\mu\|\mathrm{S}\int_\Omega \frac{1}{\|\mu\|}XY\, d\mu = \max_{i=1,2} X(\omega_i)Y(\omega_i) < X(\omega_2)Y(\omega_1),$$

on noting that $a \geqslant \max_{i=1,2}\frac{X(\omega_i)Y(\omega_i)}{2a}$. Consequently, (3) does not hold for the given μ. □

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